Hedging Options on Spot Using Futures

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Contents

1 Reason

The standard Black-Scholes model refers to European options on a spot underlier, optionally paying dividends. This means the expiry contract function has the form $\Phi(S)$, where S is the spot value of the underlier at contract expiration. Examples are options on a stock such as IBM shares or on a stock index such as Dow Jones. The problem appears when one tries to replicate the option contract by trading in the underlier. While for a stock it might be possible to actually buy or sell the underlier, for a stock index the underlier doesn't even exist physically. Therefore, traders turn to an intuitive solution: use futures on the underlier instead. These futures contracts not only exist physically but are highly liquid, making them a convenient replacement for the actual spot.

1.1 Models

The models are detailed in the **Hedging With Futures** section, here's a preview of them:

- Black Scholes, Spot: hedge using the spot underlier
- Black Scholes, Future Truncation: hedge using futures on the spot underlier, use same quantity as if hedging spot
- Black Scholes, Future Approximation: hedge using futures on the spot underlier, adjust the traded quantity starting from the spot one
- Black: use option on future model, hedge using futures on the spot underlier
- Bunea: use an own model, hedge using futures on the spot underlier

Numerical results are provided next, before the theoretical arguments, as it's immediately apparent that the **Io** method easily outperforms the alternatives.

2 Numerical Results

Here are some results obtained by running Monte-Carlo numerical simulations on the presented models. The basic input parameters are chosen such that the initial option fair value under the Black-Scholes model is about $100.

The simulations involve selling an option and replicating it by trading in the underlier. Each simulation outputs the amount of cash resulting in the bank account at expiration. Ideally, under continuous trading, it should be exactly zero. In practice due to discrete trading, it's a random number with a probability distribution. Again, ideally that distribution should be normal, with mean zero and as small as possible deviation. It's instructive to see the results for the reference **Black Scholes, Spot** model, in order to know what to expect from the other models.

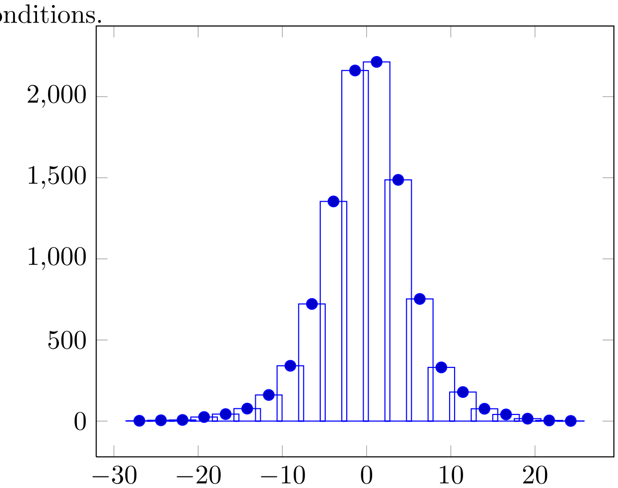
2.1 Zero Dividends

Model parameters:

- Option type: call
- Spot price: 1027.71
- Strike price: 1000
- Time to expiration: 140 days
- Interest rate: 5%
- Dividend yield: 0
- Volatility: 30%
- Hedging frequency: 140 times until expiration (thus once per day)
- Number of simulations: 10,000

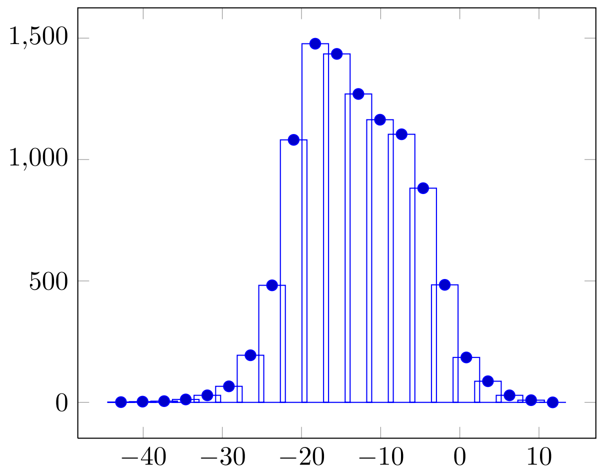
2.1.1 Black Scholes, Spot

Result mean: **-0.0392339** standard deviation: **5.32912**. Notice the need to add a margin of about 20% in order to avoid loss under moderately adverse conditions.



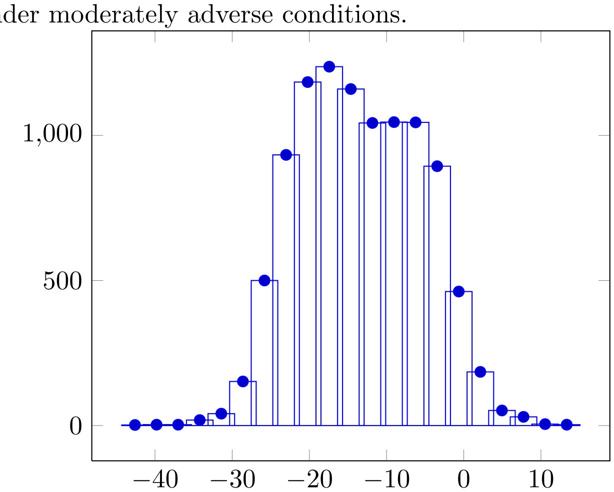
2.1.2 Black Scholes, Future Truncation

Result mean: **-13.2597** standard deviation: **6.9412**. Notice the far worse distribution of results compared to reference Black Scholes. Need to add a margin of about 30% in order to avoid loss under moderately adverse conditions.



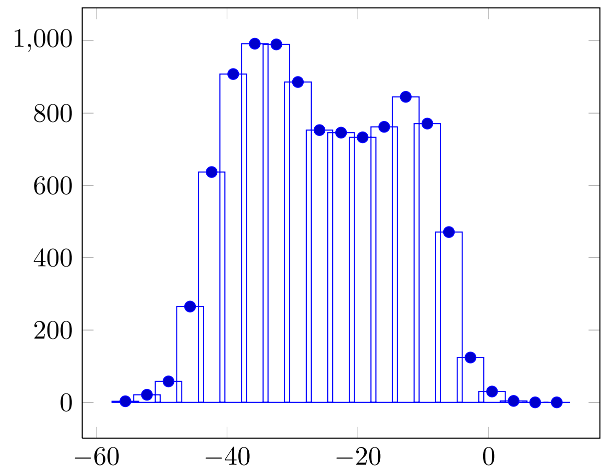
2.1.3 Black Scholes, Future Approximation

Result mean: **-13.4112** standard deviation: **7.85571**. Notice the far worse distribution of results compared to Black Scholes and no much difference compared to plain truncation. Need to add a margin of about 30% in order to avoid loss under moderately adverse conditions.



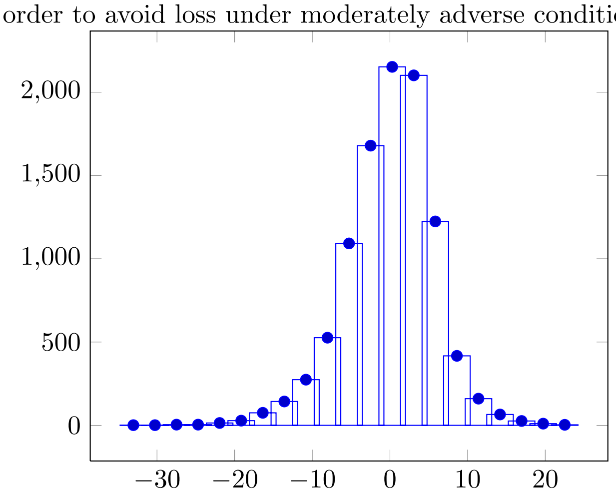
2.1.4 Black

Result mean: **-25.5818** standard deviation: **11.6996**. The method performs terribly compared to Black Scholes. Need to add a margin of about 50% in order to avoid loss under moderately adverse conditions.



2.1.5 Bunea

Result mean: **0.0120511** standard deviation: **5.58259**. The method performs virtually the same as plain Black Scholes. Need to add a margin of about 20% in order to avoid loss under moderately adverse conditions.



2.2 Non Zero Dividends

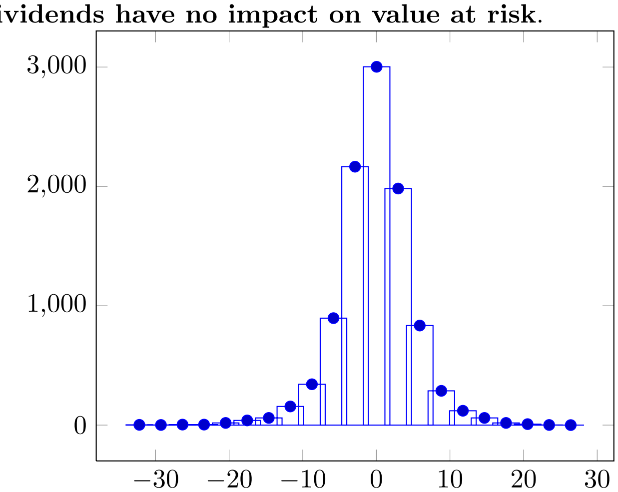
Model parameters:

- Option type: call
- Spot price: 1153.038
- Strike price: 1000
- Time to expiration: 140 days

- Interest rate: 5%
- Dividend yield: 30%
- Volatility: 30%
- Hedging frequency: 140 times until expiration (thus once per day)
- Number of simulations: 10,000

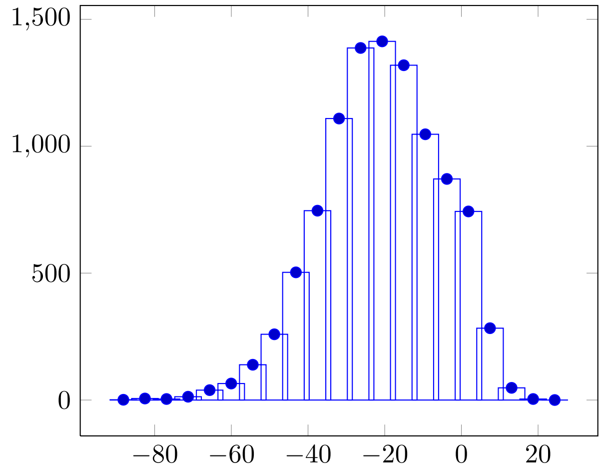
2.2.1 Black Scholes, Spot

Result mean: **-0.220088** standard deviation: **4.87253**. Notice the need to add the same margin of about 20% in order to avoid loss under moderately adverse conditions. Another thing to notice: when trading in the dividend-paying asset, **dividends have no impact on value at risk**.



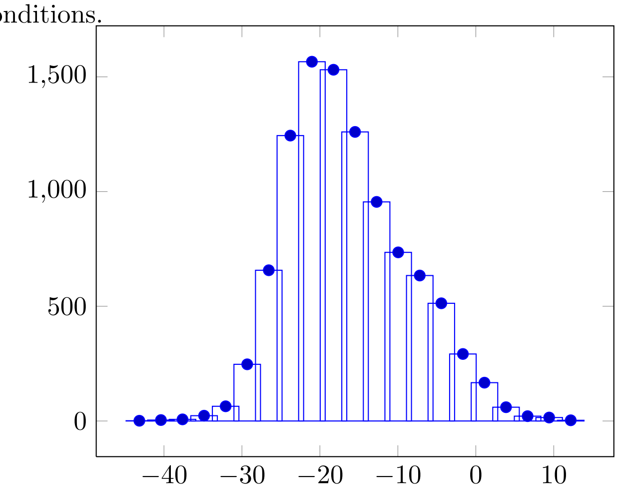
2.2.2 Black Scholes, Future Truncation

Result mean: **-20.8049** standard deviation: **15.2442**. Dividends worsen even more the performance of this hedging strategy. Need to add a margin of up to 60% in order to avoid loss under moderately adverse conditions!



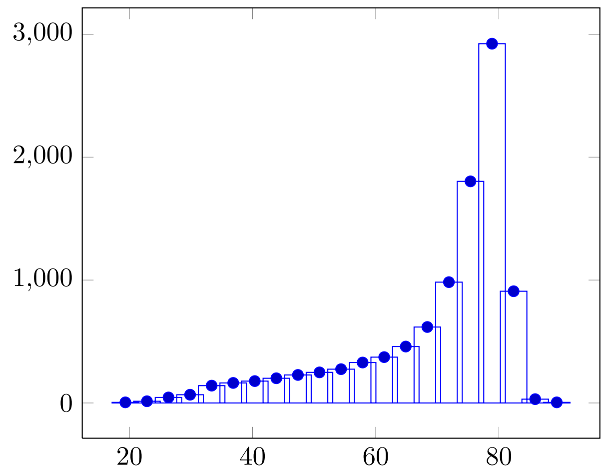
2.2.3 Black Scholes, Future Approximation

Result mean: **-16.4145** standard deviation: **7.56799**. Notice the advantage of this method compared to plain truncation: it's consistent under dividends, though not great, it doesn't worsen as the plain truncation one. Need to add the same margin of about 30% in order to avoid loss under moderately adverse conditions.



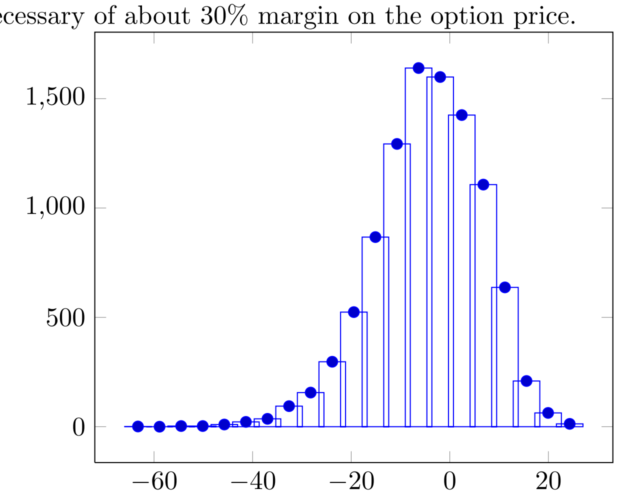
2.2.4 Black

Result mean: **69.8126** standard deviation: **12.9985**. Though terribly biased, it appears as biased towards a statistical win. Gotta study this case more, does it mean one needs to add a margin of about 80% in order to avoid loss? Or somehow, by pure gambling luck it statistically wins?



2.2.5 Bunea

Result mean: **-4.52451** standard deviation: **10.6831**. Though dividends have a worsening effect on hedging performance, the method still performs honorably, keeping a near-zero mean and consistent normal distribution of results, with a necessary of about 30% margin on the option price.



3 Some standard theory

Most of the stuff here is compiled from [1], specifically applied to the current model.

3.1 Model definition

The market consists of an optionally dividend-paying asset S whose price follows a geometric Brownian motion of growth rate μ and volatility σ. Dividends D are payed at a continuous rate q, possibly zero. And a bank account B of deterministic interest rate r. The market allows option contracts on the asset underlier, with expiration value Π for time $T >= t$ defined as some contract function Φ on the underlier asset.

The maket dynamics under the physical probability measure P is thus:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_P(t) \tag{1}$$

$$dD(t) = qS(t)dt \tag{2}$$

$$\frac{dB(t)}{B(t)} = rdt \tag{3}$$

$$\Pi(T) = \Phi(S(T)) \tag{4}$$

3.2 Pricing

Since holding the asset earns dividends, the strategy is to store them in a bank account so one defines the gain process G which reflects the value of the asset plus the earned dividends. Let $h(t)$ denote the number of units in the bank account at time t, where $h(0) = 0$. The value and dynamics of the gain process is thus:

$$G(t) = S(t) + h(t) * B(t) \tag{5}$$

$$dG(t) = dS(t) + dD(t) + h(t) * dB(t) \tag{6}$$

Differentiating (4) we get:

$$dG(t) = dS(t) + dh(t) * B(t) + h(t) * dB(t) \tag{7}$$

$$dG(t) = dS(t) + dD(t) + h(t) * dB(t) \tag{8}$$

Substracting (6) and (7) we get:

$$dh(t) * B(t) = dD(t) \Rightarrow \tag{9}$$

$$dh(t) = \frac{dD(t)}{B(t)} \Rightarrow \tag{10}$$

$$h(t) = \int_0^t \frac{dD(s)}{B(s)} \tag{11}$$

Thus we get the value of the gain process G at time t:

$$G(t) = S(t) + B(t)\int_0^t \frac{dD(s)}{B(s)} \tag{12}$$

Now we have an S-economy market model consisting of an option Π, a bank account B and a dividend-paying asset S and the price processes $[\Pi(t), B(t), G(t)]$ and form a portfolio V_S by holding quantities $[h_B(t), h_\Pi(t), h_S(t)]$ in the bank, option and dividend-paying asset. We thus have the value of the S-economy portfolio:

$$V_S(t) = h_B(t)B(t) + h_\Pi(t) * \Pi(t) + h_S(t)G(t) \tag{13}$$

Move to the normalized Z-economy market model by applying the transformation $V_Z(t) = \frac{V_S(t)}{B(t)}$. Denoting $Z(t) = \frac{G(t)}{B(t)}$, and $\Pi_Z(t) = \frac{\Pi(t)}{B(t)}$the value of the Z-economy portfolio is then:

$$\begin{aligned} V_Z(t) &= h_B(t) + h_\Pi(t) * \frac{\Pi(t)}{B(t)} + h_S(t)\frac{G(t)}{B(t)} && (14) \\ &= h_B(t) + h_\Pi(t) * \Pi_Z(t) + h_S(t)Z(t) && (15) \end{aligned}$$

We need the portfolio to be self-financing, which means the dynamics under the two economies must be:

$$\begin{aligned} dV_S(t) &= h_B(t)dB(t) + h_\Pi(t) * d\Pi(t) + h_S(t)dG(t) && (16) \\ dV_Z(t) &= h_\Pi(t) * d\Pi_Z(t) + h_S(t)dZ(t) && (17) \end{aligned}$$

From the *first fundamental theorem of asset pricing* we know the market is arbitrage-free if there exists an equivalent probability measure Q for the objective physical probability measure P such that the value of the normalized portfolio is a martingale under Q. Which means the expected value of the normalized portfolio at any future time $T > t$ is equal to it's present value. Or:

$$V_Z(t) = E^Q[V_Z(T)] \tag{18}$$

We have:

$$\begin{aligned} G(t) &= S(t) + B(t)\int_0^t \frac{dD(s)}{B(s)} && (19) \\ Z(t) &= \frac{G(t)}{B(t)} \Rightarrow && (20) \\ Z(t) &= \frac{S(t)}{B(t)} + \int_0^t \frac{dD(s)}{B(s)} && (21) \end{aligned}$$

In order to derive the dynamics of $Z(t)$, we need to apply Itō's quotient rule:

$$\frac{d(S/B)}{S/B} = \frac{dS}{S} - \frac{dB}{B} + \left(\frac{dB}{B}\right)^2 - \frac{dS}{S}\frac{dB}{B} \tag{22}$$

Putting together (1), (3) and (21) we get the dynamics under the physical probability measure P:

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= \mu dt + \sigma dW_P(t) && (23)\\
\frac{dB(t)}{B(t)} &= r dt && (24)\\
\frac{d(S/B)}{S/B} &= \frac{dS}{S} && (25)\\
&- \frac{dB}{B} && (26)\\
&+ \left(\frac{dB}{B}\right)^2 && (27)\\
&- \frac{dS}{S}\frac{dB}{B} \Rightarrow && (28)\\
\frac{d(S/B)}{S/B} &= \mu dt + \sigma dW_P(t) && (29)\\
&- r dt && (30)\\
&+ (r dt)^2 && (31)\\
&- \left[\mu dt + \sigma dW_P(t)\right] r dt \Rightarrow && (32)\\
\frac{d(S/B)}{S/B} &= (\mu - r)dt + \sigma dW_P(t) && (33)\\
&+ r^2 dt^2 + \mu r dt^2 + \sigma dW_P(t) dt && (34)
\end{aligned}$$

We know that we can ignore quadratic terms like dt^2 and $dW(t)dt$ since they are much smaller (approximately zero) compared to dt and $dW(t)$, thus we know the term (33) is approximately zero so we retain only (32) and get:

$$d\left(\frac{S(t)}{B(t)}\right) = (\mu - r)\frac{S(t)}{B(t)}dt + \sigma\frac{S(t)}{B(t)}dW_P(t) \tag{35}$$

In equation (20) we also need to derive the differential of the integral term. We have:

$$d\left(\int_0^t \frac{dD(s)}{B(s)}\right) = \frac{dD(t)}{B(t)} \tag{36}$$

Putting together (2), (34), (35) and (20) we get the dynamics of the Z process under the physical measure P:

$$\begin{aligned} dZ(t) &= (\mu - r)\frac{S(t)}{B(t)}dt + \sigma\frac{S(t)}{B(t)}dW_P(t) + \frac{dD(t)}{B(t)} && (37)\\ &= (\mu - r)\frac{S(t)}{B(t)}dt + \sigma\frac{S(t)}{B(t)}dW_P(t) + q\frac{S(t)}{B(t)}dt && (38)\\ &= (\mu - r + q)\frac{S(t)}{B(t)}dt + \sigma\frac{S(t)}{B(t)}dW_P(t) && (39) \end{aligned}$$

From (16), in order to avoid arbitrage we need to change the measure to a risk neutral one Q such that Z(t) is a martingale under Q.

We do this by applying th Girsanov theorem wich sais there exists a value λ such that we can transform between the P measure and the Q measure by the equation

$$dW_P(t) = \lambda dt + dW_Q(t) \qquad (40)$$

Plugging (39) into (38) we get the dynamics of the Z process under the Q measure:

$$\begin{aligned} dZ(t) &= (\mu - r + q)\frac{S(t)}{B(t)}dt + \sigma\frac{S(t)}{B(t)}dW_P(t) && (41)\\ &= (\mu - r + q)\frac{S(t)}{B(t)}dt + \sigma\frac{S(t)}{B(t)}\left(\lambda dt + dW_Q(t)\right) && (42)\\ &= (\mu - r + q + \sigma\lambda)\frac{S(t)}{B(t)}dt + \sigma\frac{S(t)}{B(t)}dW_Q(t) && (43) \end{aligned}$$

We need the Z process to be a martingale under this measure thus we need the drift term to dissapear so we choose the change of measure λ value accordingly:

$$\begin{aligned} (\mu - r + q + \sigma\lambda) &= 0 \Rightarrow && (44)\\ \lambda &= \frac{r - q - \mu}{\sigma} && (45) \end{aligned}$$

From (16) the dynamics of the normalized portfolio under the measure Q is:

$$\begin{aligned} dV_Z(t) &= h_\Pi(t) * d\Pi_Z(t) + h_S(t)dZ(t) && (46)\\ &= h_\Pi(t) * d\Pi_Z(t) && (47)\\ &+ h_S(t)\sigma\frac{S(t)}{B(t)}dW_Q(t) && (48) \end{aligned}$$

We know that since the term from (47) depends on no drift but only a stochastic Wiener process, it is a martingale. While for the option price Π, we *define* it to be a martingale, thus for $T > t$ we *require* it to conform to the equation:

$$\Pi_Z(t) = E^Q[\Pi_Z(T)] \tag{49}$$

Since both (46) and (47) are martingales under the probability measure Q, the entire market $V_Z(t)$ is a martingale under Q. We have thus obtained two important results:

- By the *first fundamental theorem of asset pricing*, the market $V_Z(t)$ is arbitrage free. The condition for the 1'st theorem is that a martingale measure Q exists, which we have achieved by defining the change of measure value λ in (44).
- By the *second fundamental theorem of asset pricing*, the market $V_Z(t)$ is complete, meaning the value of the option Π can be replicated by trading in the asset S and using the bank account B. The condition for the 2'nd theorem is that the martingale measure Q is unique, which is true since the λ value used in measure change is unique and equal to (44).

We now have enough info to compute the price of the option. From (48):

$$\begin{aligned}
\Pi_Z(t) &= E^Q[\Pi_Z(T)] \Rightarrow && (50)\\
\frac{\Pi(t)}{B(t)} &= E^Q\left[\frac{\Pi(T)}{B(T)}\right] \Rightarrow && (51)\\
\Pi(t) &= B(t)E^Q\left[\frac{\Pi(T)}{B(T)}\right] && (52)
\end{aligned}$$

Solving equation (3) we get $B(t) = B(0)e^{rt}$. Plugging this in (51) we get:

$$\begin{aligned}
\Pi(t) &= B(0)e^{rt}E^Q\left[\frac{\Pi(T)}{B(0)e^{rT}}\right] && (53)\\
&= \frac{B(0)e^{rt}}{B(0)e^{rT}}E^Q[\Pi(T)] && (54)\\
&= e^{r(t-T)}E^Q[\Pi(T)] && (55)\\
& && (56)
\end{aligned}$$

The expiration value of the option $\Pi(T)$ is given by it's contract function $\Phi(S(T))$. For options on spot with strike price K, the contract function is defined as:

- For a call option: $\Phi(S(T)) = max(S(T) - K, 0)$.

- For a put option: $\Phi(S(T)) = max(K - S(T), 0)$.

Therefore the arbitrage free price of the option is:

$$\Pi(t) = e^{r(t-T)} E^Q[\Phi(S(T))] \tag{57}$$

$$= e^{r(t-T)} \int_{-\infty}^{+\infty} \Phi(s) g(s) ds \tag{58}$$

where s is the possible asset price at expiration and $g(s)$ is it's probability density function. In order to obtain the probability density function we need to derive the dynamics of the asset price under the martingale probability measure Q. Putting (1), (39) and (44) together, we get:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_P(t) \tag{59}$$

$$dW_P(t) = \lambda dt + dW_Q(t) \tag{60}$$

$$\lambda = \frac{r - q - \mu}{\sigma} \tag{61}$$

So

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma \left(\frac{r - q - \mu}{\sigma} dt + dW_Q(t) \right) \tag{62}$$

$$= (\mu + r - q - \mu) dt + \sigma dW_Q(t) \tag{63}$$

$$= (r - q) dt + \sigma dW_Q(t) \tag{64}$$

A nice property of the martingale measure is that under it, we get rid of the growth rate μ. So under the martingale measure Q, the asset follows a geometric Brownian motion of dynamics:

$$\frac{dS(t)}{S(t)} = (r - q) dt + \sigma dW_Q(t) \tag{65}$$

By [2], for $T > t$ the solution of this equation is

$$S(T) = S(t) e^{(r - q - \frac{\sigma^2}{2})t + \sigma W_Q(t)} \tag{66}$$

$$= e^{\ln S(t)} e^{(r - q - \frac{\sigma^2}{2})t + \sigma W_Q(t)} \tag{67}$$

$$= e^{(lnS(t) + r - q - \frac{\sigma^2}{2})t + \sigma W_Q(t)} \tag{68}$$

$$= e^{(lnS(t) + r - q - \frac{\sigma^2}{2})t + \sigma \sqrt{t} Y} \tag{69}$$

where Y is a standard-normally distributed random variable. Denoting $a = (lnS(t) + r - q - \frac{\sigma^2}{2})t$ and $b = \sigma\sqrt{t}$ we thus have $S(t) = e^{a+bY}$, which is a log-normally distributed random variable of mean a and standard deviation b. We know it's probability density function to be:

$$g(s) = \frac{e^{-\frac{(\ln s - a)^2}{2b^2}}}{sb\sqrt{2\pi}} \tag{70}$$

Thus, the probability density function of the asset price at expiration under the martingale measure Q is

$$g(s) = \frac{e^{-\frac{(\ln s - (lnS(t) + r - q - \frac{\sigma^2}{2}))^2}{2(\sigma\sqrt{t})^2}}}{s(\sigma\sqrt{t})\sqrt{2\pi}} \tag{71}$$

so the fair price of the option is:

$$\Pi(t, S(t)) = e^{r(t-T)} \int_{-\infty}^{+\infty} \Phi(s) \frac{e^{-\frac{\left[\ln s - (lnS(t) + r - q - \frac{\sigma^2}{2})\right]^2}{2(\sigma\sqrt{t})^2}}}{s(\sigma\sqrt{t})\sqrt{2\pi}} ds \tag{72}$$

3.3 Replication

Since from (71), the option price process $\Pi(t, S(t))$ is a function of time and underlier asset, by (1) and Itō lemma we get the dynamics of the S-economy option price process under the physical measure P:

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= \mu dt + \sigma dW_P(t) \Rightarrow && (73)\\
d\Pi(t, S(t)) &= \frac{\partial \Pi(t, S(t))}{\partial t} dt && (74)\\
&+ \mu S(t) \frac{\partial \Pi(t, S(t))}{\partial s} dt && (75)\\
&+ \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 \Pi(t, S(t))}{\partial s^2} dt && (76)\\
&+ \sigma S(t) \frac{\partial \Pi(t, S(t))}{\partial s} dW_P(t) && (77)
\end{aligned}$$

Applying relations (39) and (44) to change to the martingale measure Q:

$$\begin{aligned}
dW_P(t) &= \frac{r - q - \mu}{\sigma} dt + dW_Q(t) \Rightarrow && (78)\\
d\Pi(t, S(t)) &= \frac{\partial \Pi(t, S(t))}{\partial t} dt && (79)\\
&+ \mu S(t) \frac{\partial \Pi(t, S(t))}{\partial s} dt && (80)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 \Pi(t,S(t))}{\partial s^2} dt && (81)\\
&+ \quad \sigma S(t) \frac{\partial \Pi(t,S(t))}{\partial s} \left(\frac{r-q-\mu}{\sigma} dt + dW_Q(t) \right) && (82)
\end{aligned}
$$

Again we notice we get rid of the growth rate μ so we remain with:

$$
\begin{aligned}
d\Pi(t,S(t)) &= \frac{\partial \Pi(t,S(t))}{\partial t} dt && (83)\\
&+ (r-q)S(t)\frac{\partial \Pi(t,S(t))}{\partial s} dt && (84)\\
&+ \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 \Pi(t,S(t))}{\partial s^2} dt && (85)\\
&+ \sigma S(t) \frac{\partial \Pi(t,S(t))}{\partial s} dW_Q(t) && (86)
\end{aligned}
$$

Since the option price must me a martingale under Q, the dt part mush vanish so we remain with:

$$
d\Pi(t,S(t)) = \sigma S(t) \frac{\partial \Pi(t,S(t))}{\partial s} dW_Q(t) \quad (87)
$$

We need to transform the equation to the Z-economy, so we have:

$$
\begin{aligned}
\Pi_Z(t,S(t)) &= \frac{\Pi(t,S(t))}{B(t)} \Rightarrow && (88)\\
\int_0^t d\Pi_Z(u,S(u)) &= \frac{1}{B(t)} \int_0^t d\Pi(u,S(u)) \Rightarrow && (89)\\
d\Pi_Z(u,S(u) &= \frac{d\Pi(u,S(u))}{B(t)} && (90)
\end{aligned}
$$

Plugging (89) in (86) we get:

$$
d\Pi_Z(t,S(t)) = \sigma \frac{S(t)}{B(t)} \frac{\partial \Pi(t,S(t))}{\partial s} dW_Q(t) \quad (91)
$$

Remember (14) and (45) - the value and dynamics of the normalized portfolio under the martingale measure Q and using (90):

$$
\begin{aligned}
V_Z(t) &= h_B(t) + h_\Pi(t) * \Pi_Z(t) + h_S(t)Z(t) && (92)\\
dV_Z(t) &= h_\Pi(t) * d\Pi_Z(t) + h_S(t)\sigma \frac{S(t)}{B(t)} dW_Q(t) && (93)\\
&= h_\Pi(t) * \sigma \frac{S(t)}{B(t)} \frac{\partial \Pi(t,S(t))}{\partial s} dW_Q(t) + h_S(t)\sigma \frac{S(t)}{B(t)} dW_Q(t) && (94)\\
& && (95)
\end{aligned}
$$

If we want to replicate the option using the asset and bank account, we need the normalized portfolio to satisfy the equation $V_Z(t) = 0$ at any time. From (91):

$$
\begin{aligned}
0 &= h_B(t) + h_\Pi(t) * \Pi_Z(t) + h_S(t)Z(t) && (96)\\
0 &= h_\Pi(t) * \sigma \frac{S(t)}{B(t)} \frac{\partial \Pi(t, S(t))}{\partial s} dW_Q(t) + h_S(t)\sigma \frac{S(t)}{B(t)} dW_Q(t) && (97)
\end{aligned}
$$

The value $h_\Pi(t)$ is known, being the quantity of option contracts one buys or sells. Therefore from (96) we get the quantity we need to hold in the asset underlier at any time:

$$
\begin{aligned}
h_S(t)\sigma \frac{S(t)}{B(t)} dW_Q(t) &= -h_\Pi(t) * \sigma \frac{S(t)}{B(t)} \frac{\partial \Pi(t, S(t))}{\partial s} dW_Q(t) \Rightarrow && (98)\\
h_S(t) &= -h_\Pi(t) * \frac{\partial \Pi(t, S(t))}{\partial s} && (99)
\end{aligned}
$$

4 Hedging With Futures

4.1 Approaches

I noticed several approaches of dealing with the fact of using futures instead of spot for option replication. They're all justified by the no-arbitrage link between the spot and the future price. Having the following:

- Current time t.
- A spot instrument currently trading at price $S(t)$.
- A future contract on the respective spot with expiration at time $T > t$.
- A the future instrument currently trading at price $F(t)$.
- The spot instrument paying a known dividend yield q.
- The banks offering/asking a known interest rate r.

Then in order to avoid arbitrage it can be shown that the currently traded future and spot prices need to satisfy the relation:

$$F(t) = S(t)e^{(r-q)(T-t)} \tag{100}$$

4.2 Black Scholes, Spot

Hedge using spot. This approach is actually provided here as a reference for the other ones. It is the classical Black-Scholes model and the price of an option on the spot price of an asset underlier, hedged with the respective asset was deducted to be (72). For the case of

- European call and put options.
- Current time t.
- Expiration time $T >= t$.
- Expiration contract strike K.
- The spot price of the underlier asset instrument currently trading at price $S(t)$.

the integral formula in (72) can be shown by [3] to lead to the Black-Scholes closed form valuation:

$$\Pi_C(t) = e^{-q(T-t)}S(t)N(d_1) - e^{-r(T-t)}KN(d_2) \quad (101)$$

$$\Pi_P(t) = -e^{-q(T-t)}S(t)N(-d_1) + e^{-r(T-t)}KN(-d_2) \quad (102)$$

$$d_1 = \frac{\ln\frac{S(t)}{K} + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (103)$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (104)$$

where Π_C and Π_P are the current fair price of a call and a put option respectively, and $N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{\frac{x^2}{2}}dx$ is the standard normal cumulative distribution function.

When replicating the options, use (99), therefore hold a quantity of underlier asset equal to $\frac{\partial\Pi(t,S(t))}{\partial S}$.

4.3 Black Scholes, Future Truncation

Consider spot equals futures. Use the standard Black-Scholes model for pricing the options as in paragraph 2.2.1 and when trading the underlier simply use future instead of spot. Effectively consider $S = F$. While this is obviously not mathematically correct in all cases, for small interest rate r, dividend yield q and time to maturity $T-t$, it holds pretty well. The question answered by this document is what happens when the relation doesn't hold so well.

4.4 Black Scholes, Future Approximation

Use an approximate mathematical relation. Still price the options as in paragraph 2.2.1 but adjust the number of futures contracts. There's a document I found on the Internet at [4] which gives the following relation to use for computing the number of futures contracts to use: $h_F = h_S * e^{-(r-q)(T-t)}$, where h_F are the number of futures contracts to be deduced from the number of spot contracts h_S. It also offers a mathematical justification for it. From (1) and (99), by Itō lemma we have:

$$\begin{aligned} dF(S,t) &= \left(\frac{\partial F(S,t)}{\partial t} + \mu S(t)\frac{\partial F(S,t)}{\partial s} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F(S,t)}{\partial s^2}\right) dt && (105) \\ &+ \sigma S(t)\frac{\partial F(S,t)}{\partial s} dW_P(t) && (106) \end{aligned}$$

And we know:

$$\begin{aligned} F(S,t) &= Se^{(r-q)(T-t)} \Rightarrow && (107) \\ \frac{\partial F(S,t)}{\partial t} &= -(r-q)Se^{(r-q)t} && (108) \\ \frac{\partial F(S,t)}{\partial s} &= e^{(r-q)(T-t)} && (109) \\ \frac{\partial^2 F(S,t)}{\partial s^2} &= 0 && (110) \end{aligned}$$

Replacing in the relation for $dF(S,t)$ we get:

$$\begin{aligned} dF(S,t) &= \left(-(r-q)Se^{(r-q)(T-t)} + \mu S(t)e^{(r-q)(T-t)}\right) dt && (111) \\ &+ \sigma S(t)e^{(r-q)(T-t)} dW_P(t) && (112) \\ &= -(r-q)S(t)e^{(r-q)(T-t)} dt && (113) \\ &+ e^{(r-q)(T-t)} S(t)(\mu dt + \sigma dW_P(t)) \Rightarrow && (114) \\ dF(S,t) &= -(r-q)S(t)e^{(r-q)(T-t)} dt + e^{(r-q)(T-t)} dS(t) && (115) \end{aligned}$$

Transforming to difference equation and considering $\Delta t \simeq 0$, one arrives to the approximation:

$$\begin{aligned} \Delta F(S,t) &= -(r-q)S(t)e^{(r-q)(T-t)}\Delta t + e^{(r-q)(T-t)}\Delta S(t) \Rightarrow && (116) \\ \Delta F(S,t) &\simeq e^{(r-q)(T-t)}\Delta S(t) \Rightarrow && (117) \\ \frac{\Delta F}{\Delta S} &= e^{(r-q)(T-t)} && (118) \end{aligned}$$

standard normal cumulative distribution function By (98) we know the quantity h_S to be traded in the spot underlier is the sensitivity of option price to a move in the spot price and need to find out the the quantity h_F to be traded in the future underlier, the sensitivity of option price to a move in the future price:

$$h_S = \frac{\Delta\Pi}{\Delta S} \quad (119)$$

$$h_F = \frac{\Delta\Pi}{\Delta F} \quad (120)$$

Combining (102), (103) and (104) we get:

$$\frac{\Delta F}{\Delta S} = e^{(r-q)(T-t)} \Rightarrow \quad (121)$$

$$\Delta S = e^{-(r-q)(T-t)}\Delta F \quad (122)$$

$$h_S = \frac{\Delta\Pi}{\Delta S} \Rightarrow \quad (123)$$

$$h_S = \frac{\Delta\Pi}{e^{-(r-q)(T-t)}\Delta F} \quad (124)$$

$$= e^{(r-q)(T-t)}\frac{\Delta\Pi}{\Delta F} \quad (125)$$

$$= e^{(r-q)(T-t)}h_F \Rightarrow \quad (126)$$

$$h_F = h_S e^{-(r-q)(T-t)} \quad (127)$$

4.5 Black

Use the option on future model. The Black model is used to price options on futures underlier. In this case we're dealing with options on spot but since we plan to use futures in hedging them, one might investigate what happens when we try to model and replicate them as options on future. In this case the price of the options is:

$$\Pi_C(t) = e^{-r(T-t)}F(t)N(d_1) - e^{-r(T-t)}KN(d_2) \quad (128)$$

$$\Pi_P(t) = -e^{-r(T-t)}F(t)N(-d_1) + e^{-r(T-t)}KN(-d_2) \quad (129)$$

$$d_1 = \frac{\ln\frac{F(t)}{K} + (\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (130)$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (131)$$

where $F(t)$ is the price of the future contract on the underlier asset. The relations for the Black model can be obtained from the standard Black-Scholes formula by replacing the spot price $S(t)$ with the future price $F(t)$ and setting $r = q$.

4.6 Bunea

Use my model. None of the above properly addresses the problem of hedging with futures. To do so, start with the fundamentals. As shown in section 2.2.2, relation (110), when the spot price follows the dynamics in (1) and the future price depends on it by (99), by Itō lemma we get the dynamics of the future price:

$$\begin{aligned} dF(S,t) &= -(r-q)S(t)e^{(r-q)(T-t)}dt && (132)\\ &+ e^{(r-q)(T-t)}S(t)(\mu dt + \sigma dW_P(t)) && (133)\\ &= -(r-q)F(S,t)dt + F(S,t)(\mu dt + \sigma dW_P(t)) \Rightarrow && (134)\\ \frac{dF(S,t)}{F(S,t)} &= -(r-q)dt + (\mu dt + \sigma dW_P(t)) && (135)\\ &= (\mu - (r-q))dt + \sigma dW_P(t) && (136) \end{aligned}$$

Denoting $\alpha = \mu - (r-q)$, we notice that the dynamics of the future price under the physical measure P is still a geometric Brownian motion as (1) but with growth rate α instead of μ:

$$\frac{dF(t)}{F(t)} = \alpha dt + \sigma dW_P(t) \qquad (137)$$

Therefore, we have the same market model as in section (1.1) with two noticeable differences. The first one of them is that **the future contract pays no dividends**. The second concerns the option expiration contract function. This is the same function as in (4), since we're talking about the same option on the asset underlier. We're using a future on the contract underlier to hedge it, but when expiration time T is reached, the option value is still defined in terms of the asset underlier. As long as the future contract expires at some time $T_F >= T$, we have the relations:

$$\begin{aligned} F(T) &= S(T)e^{(r-q)(T_F-T)} \Rightarrow && (138)\\ S(T) &= F(T)e^{-(r-q)(T_F-T)} && (139) \end{aligned}$$

Therefore the option expiration contract function is:

$$\Psi(F(T)) = \Phi(S(T)) = \Phi(F(T)e^{-(r-q)(T_F-T)}) \qquad (140)$$

Therefore, following (1.1) we have the following market model for pricing options on spot using a future on the respective spot:

$$\begin{aligned} \frac{dF(t)}{F(t)} &= \alpha dt + \sigma dW_P(t) && (141)\\ dD(t) &= 0 && (142)\\ \frac{dB(t)}{B(t)} &= rdt && (143)\\ \Pi(T) &= \Phi(F(T)e^{-(r-q)(T_F-T)}) && (144) \end{aligned}$$

We know we can price the options by the integral formula in (72), but again, by [3] I can show this leads to the closed form valuation:

$$\Pi_C(t) = G(t)N(d_1) - e^{-r(T-t)}KN(d_2) \tag{145}$$
$$\Pi_P(t) = G(t)N(-d_1) + e^{-r(T-t)}KN(-d_2) \tag{146}$$
$$d_1 = \frac{\ln\frac{G(t)}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{147}$$
$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{148}$$
$$G(t) = F(t)e^{-(r-q)(T_F-T)} \tag{149}$$

The relations for the Io model can be obtained from the standard Black-Scholes formula by replacing the spot price $S(t)$ with a discounted future price $G(t) = F(t)e^{-(r-q)(T_F-T)}$ and setting $q = 0$.

5 Conclusion

When hedging options with futures, the **Bunea** method is the best choice, compared with the others.

References

[1] Thomas Björk, Arbitrage Theory In Continuous Time

[2] Wikipedia, Geometric Brownian Motion

[3] Fabrice Douglas Rouah, Four Derivations of the Black-Scholes Formula

[4] Antonie A. Kotzé, Delta Hedging: Futures Versus Underlying Spot